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Affine Weyl group symmetry of the Garnier system

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Abstract

In this paper, we show that the Garnier system in n-variables has affine Weyl group symmetry of type $B^{(1)}_{n+3}$. We also formulate the τ-functions for the Garnier system (or the Schlesinger system of rank 2) on the root lattice $Q(C_{n+3})$ and show that they satisfy Toda equations, Hirota-Miwa equations and bilinear differential equations.

1 Introduction

For the sixth Painlevé equation P_{VI}, the symmetry structure is well-known [1, 5]. Furthermore, the τ-functions for P_{VI} satisfy various bilinear relations [4, 5, 6]. But such properties are not clarified completely for the Garnier system which is an extension of P_{VI} to several variables. In this paper, we show that the Garnier system in n-variables ($n \geq 2$) has affine Weyl group symmetry of type $B^{(1)}_{n+3}$. We also formulate the τ-functions for the Garnier system (or the Schlesinger system of rank 2) on the root lattice $Q(C_{n+3})$ and show that they satisfy Toda equations, Hirota-Miwa equations and bilinear differential equations.

Consider a Fuchsian differential equation on $\mathbb{P}^1(\mathbb{C})$

$$\frac{d^2y}{dz^2} + P_1(z)\,\frac{dy}{dz} + P_2(z)\,y = 0 \tag{1.1}$$

with regular singularities $z = t_1, \dots, t_n$, $t_{n+1} = 0$, $t_{n+2} = 1$, ∞, apparent singularities $z = \lambda_1, \dots, \lambda_n$ and the Riemann scheme

$$\left(\begin{array}{ccccccc} z=t_1 & \dots & z=t_{n+2} & z=\infty & z=\lambda_1 & \dots & z=\lambda_n \\ 0 & \dots & 0 & \rho & 0 & \dots & 0 \\ \theta_1 & \dots & \theta_{n+2} & \rho+\theta_{n+3}+1 & 2 & \dots & 2 \end{array}\right), \tag{1.2}$$

assuming that the Fuchs relation

$$\sum_{j=1}^{n+3} \theta_j + 2\rho = 1 \tag{1.3}$$

is satisfied. The monodromy preserving deformations of the equation (1.1) with the scheme (1.2) is described as the following completely integrable Hamiltonian system [1]:

$$\frac{\partial \lambda_j}{\partial t_i} = \frac{\partial \mathcal{K}_i}{\partial \mu_j}, \quad \frac{\partial \mu_j}{\partial t_i} = -\frac{\partial \mathcal{K}_i}{\partial \lambda_j} \quad (i, j = 1, \ldots, n), \tag{1.4}$$

where

$$\mu_j = \operatorname*{Res}_{z=\lambda_j} P_2(z)\, dz \quad (j = 1, \ldots, n) \tag{1.5}$$

and $\mathcal{K}_i$ $(i = 1, \ldots, n)$ are rational functions in λ_j, μ_j $(j = 1, \ldots, n)$ given by

$$\mathcal{K}_i = -\operatorname*{Res}_{z=t_i} P_2(z)\, dz. \tag{1.6}$$

By the canonical transformation

$$x_i = \frac{t_i}{t_i - 1}, \quad q_i = \frac{t_i \prod_{j=1}^{n}(t_i - \lambda_j)}{\prod_{j=1, j\neq i}^{n+2}(t_i - t_j)} \quad (i = 1, \ldots, n), \tag{1.7}$$

the system (1.4) is transformed into the Hamiltonian system

$$\frac{\partial q_j}{\partial x_i} = \frac{\partial K_i}{\partial p_j}, \quad \frac{\partial p_j}{\partial x_i} = -\frac{\partial K_i}{\partial q_j} \quad (i, j = 1, \ldots, n) \tag{1.8}$$

with *polynomial* Hamiltonians K_i $(i = 1, \ldots, n)$. These K_i are given explicitly by

$$\begin{aligned} x_i(x_i - 1)K_i = {} & q_i \left(\rho + \sum_{j=1}^{n} q_j p_j\right)\left(\rho + \theta_{n+3} + 1 + \sum_{j=1}^{n} q_j p_j\right) + x_i p_i (q_i p_i - \theta_i) \\ & - \sum_{j=1, j\neq i}^{n} X_{ij}\, q_i p_i (q_j p_j - \theta_j) - \sum_{j=1, j\neq i}^{n} X_{ji}\, q_i (q_j p_j - \theta_j) p_j \\ & - \sum_{j=1, j\neq i}^{n} X_{ij}^{*}\, (q_i p_i - \theta_i) p_i q_j - \sum_{j=1, j\neq i}^{n} X_{ij}\, (q_i p_i - \theta_i) q_j p_j \\ & - (x_i + 1)(q_i p_i - \theta_i) q_i p_i + (\theta_{n+2}\, x_i + \theta_{n+1} - 1) q_i p_i, \end{aligned} \tag{1.9}$$

where

$$X_{ij} = \frac{x_i(x_j-1)}{x_j-x_i}, \quad X^*_{ij} = \frac{x_i(x_i-1)}{x_i-x_j}. \tag{1.10}$$

We call the Hamiltonian system (1.8) with the Hamiltonians (1.9) the *Garnier system*.

As is known in [1], the Garnier system is derived from the Schlesinger system (of rank 2). Then the independent and dependent variables of the Garnier system are expressed as certain rational functions in the variables of the Schlesinger system. Furthermore, the τ-functions for the Garnier system can be identified with those for the Schlesinger system. Hence we first investigate symmetries and properties of the τ-functions for the Schlesinger system. After that, we apply the obtained results to the Garnier system.

In Section 2, we give the transformations of three types, permutation of the points, sign change of the exponents and Schlesinger transformation, which act on the Schlesinger system. In Section 3, we formulate the τ-functions for the Schlesinger system on the root lattice $Q(C_{n+3})$. We also present bilinear relations which are satisfied by the τ-functions. In Section 4, we show that the Garnier system has affine Weyl group symmetry of type $B^{(1)}_{n+3}$.

2 Schlesinger system

Let A_j and G_j $(j = 1, \dots, n+2)$ be matrices of dependent variables defined as

$$A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}, \quad G_j = \begin{pmatrix} -d_j & b_j \\ c_j & d_j \end{pmatrix} \begin{pmatrix} g_j & 0 \\ 0 & h_j \end{pmatrix}. \tag{2.1}$$

Consider a system of total differential equations

$$\begin{aligned} dA_j &= \sum_{i=1, i\neq j}^{n+2} [A_i, A_j]\, d\log(t_j - t_i) \quad (j = 1, \dots, n+2), \\ dG_j &= \sum_{i=1, i\neq j}^{n+2} A_i G_j \, d\log(t_j - t_i) \qquad (j = 1, \dots, n+2), \end{aligned} \tag{2.2}$$

where $t_{n+1} = 0$, $t_{n+2} = 1$ and d is an exterior differentiation with respect to $t_1, \dots, t_n$. Here we assume

(i) $\quad \det A_j = 0, \quad \mathrm{tr} A_j = \theta_j \notin \mathbb{Z} \quad (j = 1, \dots, n+2)$;

(ii) $\quad -\sum_{j=1}^{n+2} A_j = \mathrm{diag}\,(\rho,\, \rho + \theta_{N+3}), \quad \theta_{n+3} \notin \mathbb{Z}, \quad \rho = -\sum_{j=1}^{n+3} \theta_j/2.$

We call the system (2.2) the *Schlesinger system*.

Recall that the Schlesinger system is obtained as the compatibility condition for a system of linear differential equations on $\mathbb{P}^1(\mathbb{C})$

$$\frac{\partial \vec{y}}{\partial z} = \sum_{j=1}^{n+2} \frac{A_j}{z - t_j}\,\vec{y}, \quad \frac{\partial \vec{y}}{\partial t_i} = -\frac{A_i}{z - t_i}\,\vec{y} \quad (i = 1, \ldots, n), \tag{2.3}$$

where $\vec{y} = {}^t(y_1, y_2)$ is a vector of unknown functions. The matrices G_j ($j = 1, \ldots, n{+}2$) are obtained as follows. The system (2.3) has a local fundamental solution $Y = Y(z)$ of the form

$$Y = Y_j(z)\,(z - t_j)^{\theta_j E_2} \quad (j = 1, \ldots, n + 2), \tag{2.4}$$

where

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.5}$$

Here $Y_j(z)$ is a 2×2 matrix which is holomorphic at $z = t_j$, such that

$$Y_j(z)\,\Big|_{z=t_j} = G_j, \quad G_j^{-1} A_j G_j = \theta_j E_2. \tag{2.6}$$

Note that the Schlesinger system has an ambiguity for the following transformation:

$$A_j \to C^{-1} A_j C, \quad G_j \to C^{-1} G_j \quad (j = 1, \ldots, n + 2), \tag{2.7}$$

where

$$C = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} \quad (\gamma_1, \gamma_2 \in \mathbb{C}). \tag{2.8}$$

The Schlesinger system is invariant under the action of the following transformations of three types. They are associated with (1) permutation of the points $t_1, \ldots, t_{n+2}, t_{n+3} = \infty$, (2) sign change of the exponents $\theta_1, \ldots, \theta_{n+3}$, and (3) shifting of the exponents by integers (*Schlesinger transformation*). In this section, we describe these transformations.

2.1 Permutation of the points

In the following, we use the matrix notations

$$w(A_j) = \begin{pmatrix} w(a_j) & w(b_j) \\ w(c_j) & w(d_j) \end{pmatrix} \tag{2.9}$$

and

$$w(G_j) = \begin{pmatrix} -w(d_j) & w(b_j) \\ w(c_j) & w(d_j) \end{pmatrix} \begin{pmatrix} w(g_j) & 0 \\ 0 & w(h_j) \end{pmatrix} \tag{2.10}$$

for a transformation w of the dependent variables.

The action of the symmetric group $\mathfrak{S}_{n+3}$ on the set of the points $t_1, \ldots, t_n$, $t_{n+1} = 0$, $t_{n+2} = 1$, $t_{n+3} = \infty$ can be lifted to transformations of the independent and dependent variables. Denoting the adjacent transpositions by $\sigma_1 = (12), \ldots, \sigma_{n+2} = (n+2, n+3)$, we describe the action of these σ_k on the variables t_i $(i = 1, \ldots, n)$ and a_j, b_j, c_j, d_j, g_j, h_j $(j = 1, \ldots, n+2)$.

$$\sigma_k(t_i) = t_{\sigma_k(i)}, \quad \sigma_k(A_j) = A_{\sigma_k(j)}, \quad \sigma_i(G_j) = G_{\sigma_i(j)} \tag{2.11}$$

for $k = 1, \ldots, n-1$. We remark that σ_n, σ_{n+1} and σ_{n+2} are derived from Mëbius transformations on $\mathbb{P}^1(\mathbb{C})$. The transformation σ_n is derived from $z \to (z - t_n) \,/\, (1 - t_n)$:

$$\begin{aligned}
&\sigma_n(t_i) = \frac{t_i - t_n}{1 - t_n} \quad (i \neq n), \quad \sigma_n(t_n) = \frac{-t_n}{1 - t_n}, \\
&\sigma_n(A_j) = (1 - t_n)^{\theta_{n+3} E_2} A_{\sigma_n(j)} \, (1 - t_n)^{-\theta_{n+3} E_2}, \\
&\sigma_n(G_j) = (1 - t_n)^{\rho I_2 + \theta_{n+3} E_2} G_{\sigma_n(j)} \, (1 - t_n)^{\theta_{\sigma_n(j)} E_2}.
\end{aligned} \tag{2.12}$$

Similarly, the transformation σ_{n+1} is derived from $z \to 1 - z$:

$$\sigma_{n+1}(t_i) = 1 - t_i, \quad \sigma_{n+1}(A_j) = A_{\sigma_{n+1}(j)}, \quad \sigma_{n+1}(G_j) = G_{\sigma_{n+1}(j)}, \tag{2.13}$$

and the transformation σ_{n+2} is derived from $z \to 1/z$:

$$\begin{aligned}
&\sigma_{n+2}(t_i) = \frac{t_i}{t_i - 1}, \\
&\sigma_{n+2}(A_j) = G_{n+2}^{-1} A_j G_{n+2} && (j \neq n+2), \\
&\sigma_{n+2}(A_{n+2}) = \theta_{n+3} \, G_{n+2}^{-1} E_2 G_{n+2}, \\
&\sigma_{n+2}(G_j) = G_{n+2}^{-1} G_j \, (t_j - 1)^{\rho I_2 + 2\theta_j E_2} && (j \neq n+2), \\
&\sigma_{n+2}(G_{n+2}) = G_{n+2}^{-1},
\end{aligned} \tag{2.14}$$

The action of each σ_k on the parameters θ_j is given by

$$\sigma_k(\theta_j) = \theta_{\sigma_k(j)} \quad (j = 1, \ldots, n+3). \tag{2.15}$$

2.2 Sign change of the exponents

Let Y be a fundamental solution of system (2.3). Consider the gauge transformations

$$r_k(Y) = (z - t_k)^{-\theta_k} Y \quad (k = 1, \ldots, n+2), \quad r_{n+3}(Y) = WY, \tag{2.16}$$

where

$$W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.17}$$

Each r_k acts on the parameters as follows:

$$r_k(\theta_j) = (-1)^{\delta_{jk}}\theta_j \quad (j = 1, \ldots, n+3), \tag{2.18}$$

where δ_{jk} stands for the Kronecker delta, and can be lifted to a transformation of the dependent variables. We describe the action of these r_k.

$$\begin{aligned} r_k(A_j) &= A_j - \delta_{jk}\,\theta_k I_2 && (j = 1, \ldots, n+2), \\ r_k(G_j) &= (t_j - t_k)^{-\delta_{jk}\theta_k} G_j && (j = 1, \ldots, n+2) \end{aligned} \tag{2.19}$$

for $k = 1, \ldots, n+2$.

$$r_{n+3}(A_j) = W A_j W, \quad r_{n+3}(G_j) = W G_j \quad (j = 1, \ldots .n+2) \tag{2.20}$$

for $k = n+3$. Note that the independent variables t_i $(i = 1, \ldots, n)$ are invariant under the action of each r_k.

2.3 Schlesinger transformations

In this section, we construct the Schlesinger transformations by following [3]. Let L be a subset of $\mathbb{Z}^{n+3}$ defined as

$$L = \left\{ \mu = (\mu_1, \ldots, \mu_{n+3}) \in \mathbb{Z}^{n+3} \;\middle|\; \mu_1 + \ldots + \mu_{n+3} \in 2\,\mathbb{Z} \right\}. \tag{2.21}$$

Consider the gauge transformations

$$T_\mu(Y) = R_\mu Y \quad (\mu \in L), \tag{2.22}$$

where R_μ are 2×2 matrices of rational functions in z and t_i $(i = 1, \ldots, n)$, such that

$$T_\mu(\theta_j) = \theta_j + \mu_j \quad (j = 1, \ldots, n+3). \tag{2.23}$$

Then each R_μ is determined up to multiplication by a scalar matrix and the gauge transformation T_μ can be lifted to a birational transformation (called the Schlesinger transformation) of the dependent variables.

The group of the Schlesinger transformations is generated by the transformations T_k $(k = 1, \ldots, n+2)$, such that

$$T_k(\theta_j) = \theta_j + \delta_{jk} - \delta_{j\,k+1} \quad (j = 1 \ldots, n+3), \tag{2.24}$$

and T_{n+3}, such that

$$T_{n+3}(\theta_j)=\theta_j+\delta_{j\,n+2}+\delta_{j\,n+3}\quad (j=1\ldots,n+3). \tag{2.25}$$

We describe the action of these T_k on the variables $a_j,\ b_j,\ c_j,\ d_j,\ g_j,\ h_j$ $(j=1,\ldots.n+2)$.

$$\begin{aligned}
T_k(A_j)&=A_j+\frac{R_k^*A_jR_k}{(t_k-t_{k+1})(t_k-t_j)}-\frac{R_kA_jR_k^*}{(t_k-t_{k+1})(t_{k+1}-t_j)}\quad (j\neq k,k+1),\\
T_k(A_k)&=A_{k+1}-\frac{(1+\theta_k-\theta_{k+1})R_k}{t_k-t_{k+1}}-\sum_{j=1,j\neq k,k+1}^{n+2}\frac{R_k^*A_jR_k}{(t_k-t_{k+1})(t_k-t_j)},\\
T_k(A_{k+1})&=A_k+\frac{(1+\theta_k-\theta_{k+1})R_k}{t_k-t_{k+1}}+\sum_{j=1,j\neq k,k+1}^{n+2}\frac{R_kA_jR_k^*}{(t_k-t_{k+1})(t_{k+1}-t_j)},\\
T_k(G_j)&=G_j-\frac{R_kG_j}{t_{k+1}-t_j}\quad (j\neq k,k+1),\\
T_k(G_k)&=\frac{R_k^*G_k}{t_k-t_{k+1}}+\frac{G_kE_2}{t_k-t_{k+1}}+\sum_{j=1,j\neq k}^{n+2}\frac{R_k^*G_kE_1G_k^{-1}A_jG_kE_2}{(1+\theta_k)(t_k-t_{k+1})(t_k-t_j)},\\
T_k(G_{k+1})&=R_kG_{k+1}-G_{k+1}E_2+\sum_{j=1,j\neq k}^{n+2}\frac{R_kG_{k+1}E_2G_k^{-1}A_jG_kE_1}{(1-\theta_k)(t_k-t_j)},
\end{aligned} \tag{2.26}$$

where

$$R_k=\frac{-t_k+t_{k+1}}{b_ka_{k+1}+d_kb_{k+1}}\begin{pmatrix}b_k\\ d_k\end{pmatrix}\begin{pmatrix}a_{k+1} & b_{k+1}\end{pmatrix},\quad R_k^*=(t_k-t_{k+1})I_2+R_k, \tag{2.27}$$

for $k=1,\ldots,n+1$.

$$\begin{aligned}
T_{n+2}(A_{n+2})&=R_{n+2}A_{n+2}E_1+E_2A_{n+2}R_{n+2}^*+E_2R_{n+2}^*-\sum_{j=1}^{n+1}\frac{R_{n+2}A_jR_{n+2}^*}{t_j-1},\\
T_{n+2}(A_j)&=(t_j-1)E_2A_jE_1+R_{n+2}A_jE_1+E_2A_jR_{n+2}^*+\frac{R_{n+2}A_jR_{n+2}^*}{t_j-1}\\
&\qquad (j\neq n+2),\\
T_{n+2}(G_{n+2})&=R_{n+2}G_{n+2}+E_2G_{n+2}E_2+\sum_{j=1,j\neq k}^{n+2}\frac{R_{n+2}G_{n+2}E_1G_{n+2}^{-1}A_jG_{n+2}E_2}{(1+\theta_{n+2})(t_{n+2}-t_j)},\\
T_{n+2}(G_j)&=(t_j-1)E_2G_j+R_{n+2}G_j\quad (j\neq n+2),
\end{aligned} \tag{2.28}$$

where

$$R_{n+2} = \frac{1}{(1-\theta_{n+3})\, d_{n+2}} \begin{pmatrix} 1-\theta_{n+3} \\ c_\infty \end{pmatrix} \begin{pmatrix} d_{n+2} & -b_{n+2} \end{pmatrix},$$
$$R^*_{n+2} = \frac{1}{(1-\theta_{n+3})\, d_{n+2}} \begin{pmatrix} b_{n+2} \\ d_{n+2} \end{pmatrix} \begin{pmatrix} -c_\infty & 1-\theta_{n+3} \end{pmatrix} \tag{2.29}$$

and $c_\infty = \sum_{j=1}^{n+2} t_j c_j$, for $k = n+2$.

$$\begin{aligned} T_{n+3}(A_{n+2}) &= R_{n+3}A_{n+2}E_2 + E_1 A_{n+2} R^*_{n+3} + E_1 R^*_{n+3} - \sum_{j=1}^{n+1} \frac{R_{n+3}A_j R^*_{n+3}}{t_j - 1}, \\ T_{n+3}(A_j) &= (t_j - 1)E_1 A_j E_2 + R_{n+3}A_j E_2 + E_1 A_j R^*_{n+3} + \frac{R_{n+3}A_j R^*_{n+3}}{t_j - 1} \\ &\qquad (j \neq n+2), \\ T_{n+3}(G_{n+2}) &= R_{n+3}G_{n+2} + E_1 G_{n+2} E_2 + \sum_{j=1, j\neq k}^{n+2} \frac{R_{n+3}G_{n+2}E_1 G^{-1}_{n+2} A_j G_{n+2} E_2}{(1+\theta_{n+2})(t_{n+2} - t_j)}, \\ T_{n+3}(G_j) &= (t_j - 1)E_1 G_j + R_{n+3} G_j \quad (j \neq n+2), \end{aligned} \tag{2.30}$$

where

$$R_{n+3} = \frac{1}{(1+\theta_{n+3})\, b_{n+2}} \begin{pmatrix} b_\infty \\ 1+\theta_{n+3} \end{pmatrix} \begin{pmatrix} -d_{n+2} & b_{n+2} \end{pmatrix},$$
$$R^*_{n+3} = \frac{1}{(1+\theta_{n+3})\, b_{n+2}} \begin{pmatrix} b_{n+2} \\ d_{n+2} \end{pmatrix} \begin{pmatrix} 1+\theta_{n+3} & -b_\infty \end{pmatrix} \tag{2.31}$$

and $b_\infty = \sum_{j=1}^{n+2} t_j b_j$, for $k = n+3$. Note that the independent variables t_i $(i = 1, \ldots, n)$ are invariant under the action of each T_k.

Remark 2.1. *The group of the Schlesinger transformations generated by T_k $(k = 1, \ldots, n+3)$ is isomorphic to the root lattice $Q(C_{n+3})$. The commutativity between two arbitrary Schlesinger transformations is obtained from the uniqueness of the Schlesinger transformations* [3].

3 τ-Functions on the root lattice

In this Section, we formulate the τ-functions for the Schlesinger system on the root lattice $Q(C_{n+3})$. We also present the bilinear relations of three types, Toda equations, Hirota-Miwa equations and bilinear differential equations, which are satisfied by the τ-functions.

Proposition 3.1 ([3]). *For each solution of the Schlesinger system, the 1-forms*

$$\omega_\mu = \sum_{i=1}^{N} T_\mu(H_i)\, dt_i \quad (\mu \in L) \tag{3.1}$$

are closed. Here we let

$$H_i = \sum_{j=1, j\neq i}^{n+2} \frac{1}{t_i - t_j} \left(\mathrm{tr} A_i A_j + C_{ij}\right) \quad (i = 1, \ldots, n), \tag{3.2}$$

where

$$C_{ij} = -\frac{1}{2}\, \theta_i \theta_j + \frac{\theta_i^2 + \theta_j^2}{2\,(n+1)} - \frac{\sum_{i=1}^{n+3} \theta_i^2}{2\,(n+1)(n+2)}. \tag{3.3}$$

Proposition 3.1 allows us to define a family of τ-functions by

$$d \log \tau_\mu = \omega_\mu \quad (\mu \in L), \tag{3.4}$$

up to multiplicative constants.

We also define the action of the transformations σ_k, r_l and T_μ on the τ-functions, so that it is consistent with the action of them on H_i which we call Hamiltonians. For each $\mu, \nu \in L$, the action of T_μ on τ_ν is defined by

$$T_\mu(\tau_\nu) = \tau_{\mu+\nu} \tag{3.5}$$

and the action of σ_k, r_l on τ_ν is defined by

$$\begin{aligned} \sigma_k(\tau_\nu) &= \tau_{\sigma_k(\nu)} \quad (k = 1, \ldots, n+2), \\ r_l(\tau_\nu) &= \tau_{r_l(\nu)} \quad\ \ (l = 1, \ldots, n+3), \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} \sigma_k(\nu) &= (\nu_{\sigma_k(1)}, \ldots, \nu_{\sigma_k(n+3)}), \\ r_l(\nu) &= (\nu_1, \ldots, \nu_{l-1}, -\nu_l, \nu_{l+1}, \ldots, \nu_{n+3}). \end{aligned} \tag{3.7}$$

In Section 3.1, we describe the action of the transformations σ_k, r_l and T_μ on the Hamiltonians, which is obtained from the action of them on the independent and dependent variables.

3.1 Symmetries for Hamiltonians

We first describe the action of the Schlesinger transformation T_μ on the Hamiltonians for each $\mu \in L$ with

$$\mu_1^2 + \ldots + \mu_{n+3}^2 = 2. \tag{3.8}$$

Set

$$T_{k,l} = T_{\mathbf{e}_k+\mathbf{e}_l}, \quad T_{k,-l} = T_{\mathbf{e}_k-\mathbf{e}_l}, \quad T_{-k,-l} = T_{-\mathbf{e}_k-\mathbf{e}_l} \qquad (k,l = 1,\ldots,n+3,\ k \neq l), \tag{3.9}$$

where

$$\begin{aligned} \mathbf{e}_1 &= (1,0,0,\ldots,0,0), \\ \mathbf{e}_2 &= (0,1,0,\ldots,0,0), \\ &\vdots \\ \mathbf{e}_{n+3} &= (0,0,0,\ldots,0,1). \end{aligned} \tag{3.10}$$

We remark

$$T_k = T_{k,-(k+1)} \quad (k = 1,\ldots,n+2), \quad T_{n+3} = T_{n+2,n+3} \tag{3.11}$$

and that they act on θ_j $(j = 1,\ldots,n+3)$ as follows:

$$\begin{aligned} T_{k,l}(\theta_j) &= \theta_j + \delta_{jk} + \delta_{jl}, \\ T_{k,-l}(\theta_j) &= \theta_j + \delta_{jk} - \delta_{jl}, \\ T_{-k,-l}(\theta_j) &= \theta_j - \delta_{jk} - \delta_{jl}. \end{aligned} \tag{3.12}$$

Then the action of them on the Hamiltonians H_i $(i = 1,\ldots,n)$ is described as follows.

$$\begin{aligned} T_{k,l}(H_i) &= H_i - \frac{\mathrm{tr} A_i R_{k,l}}{(t_i - t_k)(t_i - t_l)} + \frac{\Gamma_k^i}{t_i - t_k} + \frac{\Gamma_l^{-i}}{t_i - t_l} + \sum_{j=1, j\neq i}^{n+2} \frac{\Gamma_{k,l}}{t_i - t_j} \qquad (i \neq k,l), \\ T_{k,l}(H_k) &= H_k - \sum_{j=1, j\neq k,l}^{n+2} \frac{\mathrm{tr} A_i R_{k,l}}{(t_k - t_j)(t_k - t_l)} - \frac{(n-1)(1+\theta_k+\theta_l)}{2\,(n+1)(t_k - t_l)} \\ &\quad + \sum_{j=1, j\neq k,l}^{n+2} \frac{\Gamma_k^j}{t_k - t_j} + \sum_{j=1, j\neq k}^{n+2} \frac{\Gamma_{k,l}}{t_k - t_j}, \\ T_{k,l}(H_l) &= H_l - \sum_{j=1, j\neq k,l}^{n+2} \frac{\mathrm{tr} A_i R_{k,l}}{(t_l - t_j)(t_l - t_k)} - \frac{(n-1)(1+\theta_k+\theta_l)}{2\,(n+1)(t_l - t_k)} \\ &\quad + \sum_{j=1, j\neq k,l}^{n+2} \frac{\Gamma_k^{-j}}{t_l - t_j} + \sum_{j=1, j\neq l}^{n+2} \frac{\Gamma_{k,l}}{t_l - t_j}, \end{aligned} \tag{3.13}$$

where

$$
\begin{aligned}
&\mathit{\Gamma}_k^j = -\frac{\theta_j}{2} + \frac{1 + 2\,\theta_k}{2\,(n+2)}, \qquad \mathit{\Gamma}_k^{-j} = \frac{\theta_j}{2} + \frac{1 + 2\,\theta_k}{2\,(n+2)}, \\
&\mathit{\Gamma}_{k,l} = -\frac{1 + \theta_k + \theta_l}{(n+1)(n+2)}, \quad R_{k,l} = \frac{t_k - t_l}{b_k d_l - d_k b_l} \begin{pmatrix} b_k \\ d_k \end{pmatrix} \begin{pmatrix} -d_l & b_l \end{pmatrix},
\end{aligned} \tag{3.14}
$$

for $k, l = 1, \ldots, n+2$ with $k \neq l$.

$$
\begin{aligned}
T_{k,-l}(H_i) &= H_i - \frac{\mathrm{tr} A_i R_{k,-l}}{(t_i - t_k)(t_i - t_l)} + \frac{\mathit{\Gamma}_k^i}{t_i - t_k} + \frac{\mathit{\Gamma}_{-l}^{-i}}{t_i - t_l} + \sum_{j=1, j\neq i}^{n+2} \frac{\mathit{\Gamma}_{k,-l}}{t_i - t_j} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i \neq k, l), \\
T_{k,-l}(H_k) &= H_k - \sum_{j=1, j\neq k,l}^{n+2} \frac{\mathrm{tr} A_i R_{k,-l}}{(t_k - t_j)(t_k - t_l)} - \frac{(n-1)(1 + \theta_k - \theta_l)}{2\,(n+1)(t_k - t_l)} \\
&\qquad + \sum_{j=1, j\neq k,l}^{n+2} \frac{\mathit{\Gamma}_k^j}{t_k - t_j} + \sum_{j=1, j\neq k}^{n+2} \frac{\mathit{\Gamma}_{k,-l}}{t_k - t_j}, \\
T_{k,-l}(H_l) &= H_l - \sum_{j=1, j\neq k,l}^{n+2} \frac{\mathrm{tr} A_i R_{k,-l}}{(t_l - t_j)(t_l - t_k)} - \frac{(n-1)(1 + \theta_k - \theta_l)}{2\,(n+1)(t_l - t_k)} \\
&\qquad + \sum_{j=1, j\neq k,l}^{n+2} \frac{\mathit{\Gamma}_{-l}^{-j}}{t_l - t_j} + \sum_{j=1, j\neq l}^{n+2} \frac{\mathit{\Gamma}_{k,-l}}{t_l - t_j},
\end{aligned} \tag{3.15}
$$

where

$$
\begin{aligned}
&\mathit{\Gamma}_{k,-l} = -\frac{1 + \theta_k - \theta_l}{(n+1)(n+2)}, \qquad \mathit{\Gamma}_{-k}^{-j} = \frac{\theta_j}{2} + \frac{1 - 2\,\theta_k}{2\,(n+2)}, \\
&R_{k,-l} = \frac{-t_k + t_l}{b_k a_l + d_k b_l} \begin{pmatrix} b_k \\ d_k \end{pmatrix} \begin{pmatrix} a_l & b_l \end{pmatrix},
\end{aligned} \tag{3.16}
$$

for $k, l = 1, \ldots, n+2$ with $k \neq l$.

$$\begin{aligned}
T_{-k,-l}(H_i) &= H_i - \frac{\mathrm{tr} A_i R_{-k,-l}}{(t_i - t_k)(t_i - t_l)} + \frac{\Gamma^i_{-k}}{t_i - t_k} + \frac{\Gamma^{-i}_{-l}}{t_i - t_l} + \sum_{j=1, j\neq i}^{n+2} \frac{\Gamma_{-k,-l}}{t_i - t_j} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i \neq k, l), \\
T_{-k,-l}(H_k) &= H_k - \sum_{j=1, j\neq k,l}^{n+2} \frac{\mathrm{tr} A_i R_{-k,-l}}{(t_k - t_j)(t_k - t_l)} - \frac{(n-1)(1-\theta_k-\theta_l)}{2\,(n+1)(t_k - t_l)} \\
&\qquad + \sum_{j=1, j\neq k,l}^{n+2} \frac{\Gamma^j_{-k}}{t_k - t_j} + \sum_{j=1, j\neq k}^{n+2} \frac{\Gamma_{-k,-l}}{t_k - t_j}, \\
T_{-k,-l}(H_l) &= H_l - \sum_{j=1, j\neq k,l}^{n+2} \frac{\mathrm{tr} A_i R_{-k,-l}}{(t_l - t_j)(t_l - t_k)} - \frac{(n-1)(1-\theta_k-\theta_l)}{2\,(n+1)(t_l - t_k)} \\
&\qquad + \sum_{j=1, j\neq k,l}^{n+2} \frac{\Gamma^{-j}_{-l}}{t_l - t_j} + \sum_{j=1, j\neq l}^{n+2} \frac{\Gamma_{-k,-l}}{t_l - t_j},
\end{aligned} \tag{3.17}$$

where

$$\begin{aligned}
&\Gamma_{-k,-l} = -\frac{1-\theta_k-\theta_l}{(n+1)(n+2)}, \quad \Gamma^j_{-k} = -\frac{\theta_j}{2} + \frac{1-2\,\theta_k}{2\,(n+2)}, \\
&R_{-k,-l} = \frac{t_k - t_l}{a_k b_l - b_k a_l} \begin{pmatrix} b_k \\ -a_k \end{pmatrix} \begin{pmatrix} a_l & b_l \end{pmatrix},
\end{aligned} \tag{3.18}$$

for $k, l = 1, \ldots, n+2$ with $k \neq l$.

$$\begin{aligned}
T_{k,n+3}(H_i) &= H_i + \frac{1}{t_i - t_k}\left(a_i + b_i\,\frac{d_k}{b_k}\right) + \frac{\Gamma^i_k}{t_i - t_k} + \sum_{j=1, j\neq i}^{n+2} \frac{\Gamma_{k,n+3}}{t_i - t_j} \quad (i \neq k), \\
T_{k,n+3}(H_k) &= H_k + \sum_{j=1, j\neq k}^{n+2} \frac{1}{t_k - t_j}\left(a_j + b_j\,\frac{d_k}{b_k}\right) + \sum_{j=1, j\neq k}^{n+2} \frac{\Gamma^j_k + \Gamma_{k,n+3}}{t_k - t_j}
\end{aligned} \tag{3.19}$$

for $k = 1, \dots, n+2$.

$$T_{k,-(n+3)}(H_i) = H_i + \frac{1}{t_i - t_k}\left(d_i + c_i\,\frac{a_k}{c_k}\right) + \frac{\varGamma^i_k}{t_i - t_k} + \sum_{j=1, j\neq i}^{n+2} \frac{\varGamma_{k,-(n+3)}}{t_i - t_j} \qquad (i \neq k),$$

$$T_{k,-(n+3)}(H_k) = H_k + \sum_{j=1, j\neq k}^{n+2} \frac{1}{t_k - t_j}\left(d_j + c_j\,\frac{a_k}{c_k}\right) + \sum_{j=1, j\neq k}^{n+2} \frac{\varGamma^j_k + \varGamma_{k,-(n+3)}}{t_k - t_j} \tag{3.20}$$

for $k = 1, \dots, n+2$.

$$T_{n+3,-k}(H_i) = H_i + \frac{1}{t_i - t_k}\left(a_i - b_i\,\frac{a_k}{b_k}\right) + \frac{\varGamma^i_{-k}}{t_i - t_k} + \sum_{j=1, j\neq i}^{n+2} \frac{\varGamma_{n+3,-k}}{t_i - t_j} \qquad (i \neq k),$$

$$T_{n+3,-k}(H_k) = H_k + \sum_{j=1, j\neq k}^{n+2} \frac{1}{t_k - t_j}\left(a_j - b_j\,\frac{a_k}{b_k}\right) + \sum_{j=1, j\neq k}^{n+2} \frac{\varGamma^j_{-k} + \varGamma_{n+3,-k}}{t_k - t_j} \tag{3.21}$$

for $k = 1, \dots, n+2$.

$$T_{-k,-(n+3)}(H_i) = H_i + \frac{1}{t_i - t_k}\left(d_i - c_i\,\frac{d_k}{c_k}\right) + \frac{\varGamma^i_{-k}}{t_i - t_k} + \sum_{j=1, j\neq i}^{n+2} \frac{\varGamma_{-k,-(n+3)}}{t_i - t_j} \qquad (i \neq k),$$

$$T_{-k,-(n+3)}(H_k) = H_k + \sum_{j=1, j\neq k}^{n+2} \frac{1}{t_k - t_j}\left(d_j - c_j\,\frac{d_k}{c_k}\right) + \sum_{j=1, j\neq k}^{n+2} \frac{\varGamma^j_{-k} + \varGamma_{-k,-(n+3)}}{t_k - t_j} \tag{3.22}$$

for $k = 1, \dots, n+2$. For the other $\mu \in L$, the action of T_μ on the Hamiltonians, which is not described in this paper, is similarly obtained from its action on the dependent variables.

Next we describe the action of the transformations σ_k $(k = 1, \dots, n+2)$ and r_l $(l = 1, \dots, n+3)$ on the Hamiltonians. Since H_i $(i = 1, \dots, n)$ are invariant under the action of each σ_k and r_l, we obtain

$$\sigma_k T_\mu(H_i) = T_{\sigma_k(\mu)}(H_i), \quad r_l T_\mu(H_i) = T_{r_l(\mu)}(H_i) \quad (\mu \in L), \tag{3.23}$$

where

$$\begin{aligned} \sigma_k(\mu) &= (\mu_{\sigma_k(1)}, \dots, \mu_{\sigma_k(n+3)}), \\ r_l(\mu) &= (\mu_1, \dots, \mu_{l-1}, -\mu_l, \mu_{l+1}, \dots, \mu_{n+3}). \end{aligned} \tag{3.24}$$

3.2 Toda equations

In this section, we present the Toda equations for the Schlesinger transformations T_k $(k = 1, \dots, n+3)$. Set

$$\widetilde{H}_i = H_i - \sum_{j=1, j\neq i}^{n+2} \frac{C_{ij}}{t_i - t_j} = \sum_{j=1, j\neq i}^{n+2} \frac{\mathrm{tr} A_i A_j}{t_i - t_j} \quad (i = 1, \dots, n). \tag{3.25}$$

Then we have

Theorem 3.2 ([3]). *The Hamiltonians $\widetilde{H}_i$ $(i = 1, \dots, n)$ satisfy the following equations*:

$$\begin{aligned} T_k(\widetilde{H}_i) + T_k^{-1}(\widetilde{H}_i) - 2\,\widetilde{H}_i &= \partial_{t_i} \log \frac{(G_{k+1}^{-1} G_k)_{22} (G_k^{-1} G_{k+1})_{22}}{(t_k - t_{k+1})^2} \\ &\qquad\qquad (k = 1, \dots, n+1), \\ T_{n+2}(\widetilde{H}_i) + T_{n+2}^{-1}(\widetilde{H}_i) - 2\,\widetilde{H}_i &= \partial_{t_i} \log (G_{n+2})_{22} (G_{n+2}^{-1})_{22}, \\ T_{n+3}(\widetilde{H}_i) + T_{n+3}^{-1}(\widetilde{H}_i) - 2\,\widetilde{H}_i &= \partial_{t_i} \log (G_{n+2})_{12} (G_{n+2}^{-1})_{21}, \end{aligned} \tag{3.26}$$

where $(G_j)_{kl}$ stands for the (k, l)-component of the 2×2 matrix G_j.

We also obtain the following lemma.

Lemma 3.3. *The Hamiltonians $\widetilde{H}_i$ $(i = 1, \dots, n)$ satisfy the following equations*:

$$\begin{aligned} \partial_{t_k}(\widetilde{H}_{k+1}) &= \frac{\mathrm{tr} A_k A_{k+1}}{(t_k - t_{k+1})^2} \quad (k = 1, \dots, n-1), \\ \partial_{t_n} \left(\sum_{i=1}^{n} (t_i - 1) \widetilde{H}_i \right) &= \frac{\mathrm{tr} A_n A_{n+1}}{t_n^2}, \\ (\delta^* + 1) \left(\sum_{i=1}^{n} t_i \widetilde{H}_i \right) &= -\mathrm{tr} A_{n+1} A_{n+2} - \frac{1}{2} \sum_{i=1}^{n+2} \sum_{j=1, j\neq i}^{n+2} C_{ij}, \\ (\delta + 1) \left(\sum_{i=1}^{n} t_i \widetilde{H}_i \right) &= \theta_{n+3}\, d_{n+2} + \theta_{n+2}(\rho + \theta_{n+2}) - \frac{1}{2} \sum_{i=1}^{n+2} \sum_{j=1, j\neq i}^{n+2} C_{ij}, \end{aligned} \tag{3.27}$$

where $\partial_i = \partial / \partial t_i$ and

$$\delta = \sum_{i=1}^{n} t_i (t_i - 1)\, \partial_{t_i}, \quad \delta^* = \sum_{i=1}^{n} (t_i - 1)\, \partial_{t_i}. \tag{3.28}$$

Proof The first equation of (3.27) is obtained by a direct computation. The second equation of (3.27) is obtained by using

$$\sum_{i=1}^{n}(t_i-1)\widetilde{H}_i=-\sum_{j=1,j\neq n+1}^{n+2}\frac{\mathrm{tr}A_jA_{n+1}}{t_j}+\sum_{i=1}^{n+2}\sum_{j=1,j\neq i}^{n+2}\mathrm{tr}A_iA_j \tag{3.29}$$

and

$$\sum_{i=1}^{n+2}\sum_{j=1,j\neq i}^{n+2}\mathrm{tr}A_iA_j=-\sum_{i=1}^{n+2}\sum_{j=1,j\neq i}^{n+2}C_{ij}. \tag{3.30}$$

The third equation of (3.27) is obtained by using (3.30),

$$\sum_{i=1}^{n}t_i\widetilde{H}_i=\sum_{j=1}^{n+1}\frac{\mathrm{tr}A_jA_{n+2}}{t_j-1}+\frac{1}{2}\sum_{i=1}^{n+2}\sum_{j=1,j\neq i}^{n+2}\mathrm{tr}A_iA_j \tag{3.31}$$

and

$$(\delta^*+1)\left(\sum_{j=1}^{n+1}\frac{\mathrm{tr}A_jA_{n+2}}{t_j-1}\right)=-\mathrm{tr}A_{n+1}A_{n+2}. \tag{3.32}$$

The fourth equation of (3.27) is obtained by using (3.30), (3.31) and

$$(\delta+1)\left(\sum_{j=1}^{n+1}\frac{\mathrm{tr}A_jA_{n+2}}{t_j-1}\right)=\theta_{n+3}\,d_{n+2}+\theta_{n+2}(\rho+\theta_{n+2}). \tag{3.33}$$

□

From Theorem 3.2, Lemma 3.3 and the following identities:

$$\begin{aligned}(G_{k+1}^{-1}G_k)_{22}(G_k^{-1}G_{k+1})_{22}&=-\frac{\mathrm{tr}A_kA_{k+1}}{\theta_k\theta_{k+1}}\quad(k=1,\ldots,n+1),\\(G_{n+2})_{22}(G_{n+2}^{-1})_{22}&=\frac{d_{n+2}}{\theta_{n+2}},\\(G_{n+2})_{12}(G_{n+2}^{-1})_{21}&=\frac{a_{n+2}}{\theta_{n+2}},\end{aligned} \tag{3.34}$$

we obtain

$$T_k(\widetilde{H}_i)+T_k^{-1}(\widetilde{H}_i)-2\,\widetilde{H}_i=\partial_{t_i}\log X_k\quad(k=1,\ldots,n+3), \tag{3.35}$$

where

$$
\begin{aligned}
X_k &= \partial_{t_k}(\widetilde{H}_{k+1}) \quad (k=1,\ldots,n-1),\\
X_n &= \partial_{t_n}\left(\sum_{i=1}^{n}(t_i-1)\widetilde{H}_i\right),\\
X_{n+1} &= (\delta^*+1)\left(\sum_{i=1}^{n} t_i\widetilde{H}_i\right)+\frac{1}{2}\sum_{i=1}^{n+2}\sum_{j=1,j\neq i}^{n+2} C_{ij},\\
X_{n+2} &= (\delta+1)\left(\sum_{i=1}^{n} t_i\widetilde{H}_i\right)+\frac{1}{2}\sum_{i=1}^{n+2}\sum_{j=1,j\neq i}^{n+2} C_{ij}-\theta_{n+2}(\rho+\theta_{n+2}),\\
X_{n+3} &= (\delta+1)\left(\sum_{i=1}^{n} t_i\widetilde{H}_i\right)+\frac{1}{2}\sum_{i=1}^{n+2}\sum_{j=1,j\neq i}^{n+2} C_{ij}-\theta_{n+2}(\rho+\theta_{n+2}+\theta_{n+3}).
\end{aligned} \tag{3.36}
$$

Here we introduce the Hirota derivatives D_i $(i=1,\cdots,n)$ defined by

$$
P(D_1,\cdots,D_n)\,\varphi\cdot\psi = P(\partial_{t_1},\cdots,\partial_{t_n})\,(\varphi(s+t)\psi(s-t))\,\big|_{t=0}, \tag{3.37}
$$

where $P(D_1,\cdots,D_n)$ is a polynomial in the derivations D_i $(i=1,\ldots,n)$. By the definition, we obtain

$$
\begin{aligned}
D_i\,\varphi\cdot\psi &= \partial_{t_i}(\varphi)\,\psi-\varphi\,\partial_{t_i}(\psi),\\
D_iD_j\,\varphi\cdot\psi &= \partial_{t_i}\partial_{t_j}(\varphi)\,\psi-\partial_{t_i}(\varphi)\,\partial_{t_j}(\psi)-\partial_{t_j}(\varphi)\,\partial_{t_i}(\psi)+\psi\,\partial_{t_i}\partial_{t_j}(\varphi)
\end{aligned} \tag{3.38}
$$

and

$$
\begin{aligned}
\partial_{t_i}\log\frac{\varphi}{\psi} &= \frac{D_i\,\varphi\cdot\psi}{\varphi\cdot\psi},\\
\partial_{t_i}\partial_{t_j}\log\varphi\psi &= \frac{D_iD_j\,\varphi\cdot\psi}{\varphi\cdot\psi}-\frac{D_i\,\varphi\cdot\psi}{\varphi\cdot\psi}\frac{D_j\,\varphi\cdot\psi}{\varphi\cdot\psi}.
\end{aligned} \tag{3.39}
$$

By substituting (3.25) into (3.35), we obtain the Toda and Toda-like equations expressed in terms of the Hirota derivatives.

Theorem 3.4. *For the Schlesinger transformations T_k $(k=1,\ldots,n+3)$,*

we have the following Toda and Toda-like equations:

$$
\begin{aligned}
F_k T_k(\tau_0)\, T_k^{-1}(\tau_0) &= D_k D_{k+1} \tau_0 \cdot \tau_0 - \frac{2\, C_{k\,k+1}}{(t_k - t_{k+1})^2}\, \tau_0^2 \quad (k = 1, \cdots, n-1), \\
F_n T_n(\tau_0)\, T_n^{-1}(\tau_0) &= \sum_{i=1}^{n} (t_i - 1) D_i D_n \tau_0 \cdot \tau_0 + 2\, \partial_{t_n}(\tau_0) \cdot \tau_0 - \frac{2\, C_{n\,n+1}}{t_n^2}\, \tau_0^2, \\
F_{n+1} T_{n+1}(\tau_0)\, T_{n+1}^{-1}(\tau_0) &= \sum_{i=1}^{n} \sum_{j=1}^{n} (t_i - 1) t_j D_i D_j \tau_0 \cdot \tau_0 \\
&\quad + 2 \sum_{i=1}^{n} (2\, t_i - 1)\, \partial_{t_i}(\tau_0) \cdot \tau_0 + 2\, C_{n+1\,n+2}\, \tau_0^2, \\
F_{n+2} T_{n+2}(\tau_0)\, T_{n+2}^{-1}(\tau_0) &= \sum_{i=1}^{n} \sum_{j=1}^{n} t_i (t_i - 1) t_j D_i D_j\, \tau_0 \cdot \tau_0 + 2 \sum_{i=1}^{n} t_i^2\, \partial_{t_i}(\tau_0) \cdot \tau_0 \\
&\quad + 2 \left\{ \theta_{n+2}(\rho + \theta_{n+2}) + \sum_{j=1}^{n+1} C_{i\,n+2} \right\} \tau_0^2, \\
F_{n+3} T_{n+3}(\tau_0)\, T_{n+3}^{-1}(\tau_0) &= \sum_{i=1}^{n} \sum_{j=1}^{n} t_i (t_i - 1) t_j D_i D_j \tau_0 \cdot \tau_0 + 2 \sum_{i=1}^{n} t_i^2\, \partial_{t_i}(\tau_0) \cdot \tau_0 \\
&\quad + 2 \left\{ \theta_{n+2}(\rho + \theta_{n+2} + \theta_{n+3}) + \sum_{j=1}^{n+1} C_{i\,n+2} \right\} \tau_0^2,
\end{aligned} \tag{3.40}
$$

where

$$
\begin{aligned}
F_k &= (t_k - t_{k+1})^{-1/2} \prod_{j=1, j\neq k}^{n+2} (t_k - t_j)^{-\Gamma_k^j} \prod_{j=1, j\neq k+1}^{n+2} (t_{k+1} - t_j)^{-\Gamma_{-k+1}^{-j}} \\
&\qquad \times \prod_{i=1}^{n+2} \prod_{j=1, j\neq i}^{n+2} (t_i - t_j)^{-\Gamma_{k,-(k+1)}/2} \quad (k = 1, \ldots, n+1), \\
F_{n+2} &= \prod_{j=1}^{n+1} (t_j - 1)^{-\Gamma_{n+2}^j} \prod_{i=1}^{n+2} \prod_{j=1, j\neq i}^{n+2} (t_i - t_j)^{-\Gamma_{n+2,-(n+3)}/2}, \\
F_{n+3} &= \prod_{j=1}^{n+1} (t_j - 1)^{-\Gamma_{n+2}^j} \prod_{i=1}^{n+2} \prod_{j=1, j\neq i}^{n+2} (t_i - t_j)^{-\Gamma_{n+2,n+3}/2}.
\end{aligned} \tag{3.41}
$$

We note that the Toda equation for T_{n+1} is equivalent to the equation given in [8].

3.3 Hirota-Miwa equations

In the following, we set

$$\tau_{k,l} = T_{k,l}(\tau_0), \quad \tau_{k,-l} = T_{k,-l}(\tau_0) \quad (k,l = 1,\dots,n+3,\ k \neq l). \tag{3.42}$$

We first present the Hirota-Miwa equation for the following six τ-functions:

$$\tau_{n+2,n+3}, \quad \tau_{n+1,n+2}, \quad \tau_{n+2,-(n+1)}, \quad \tau_{n+1,n+3}, \quad \tau_{n+3,-(n+1)}, \quad \tau_0.$$

The action of transformations $T_{n+1,n+2}$, $T_{n+3,-(n+1)}$ and $T_{n+2,n+3}$ on the Hamiltonians H_i $(i = 1,\dots,n)$ is described as follows:

$$\begin{aligned}
T_{n+1,n+2}(H_i) &= H_i - \frac{\mathrm{tr} A_i R_{n+1,n+2}}{t_i(t_i-1)} + \frac{\mathit{\Gamma}^i_{n+1}}{t_i} + \frac{\mathit{\Gamma}^{-i}_{n+2}}{t_i-1} + \sum_{j=1,j\neq i}^{n+2} \frac{\mathit{\Gamma}_{n+1,n+2}}{t_i-t_j}, \\
T_{n+3,-(n+1)}(H_i) &= H_i + \frac{1}{t_i}\left(a_i - b_i\frac{a_{n+1}}{b_{n+1}}\right) + \frac{\mathit{\Gamma}^i_{-(n+1)}}{t_i} + \sum_{j=1,j\neq i}^{n+2} \frac{\mathit{\Gamma}_{n+3,-(n+1)}}{t_i-t_j}, \\
T_{n+2,n+3}(H_i) &= H_i + \frac{1}{t_i-1}\left(a_i + b_i\frac{d_{n+2}}{b_{n+2}}\right) + \frac{\mathit{\Gamma}^i_{n+2}}{t_i-1} + \sum_{j=1,j\neq i}^{n+2} \frac{\mathit{\Gamma}_{n+2,n+3}}{t_i-t_j}.
\end{aligned} \tag{3.43}$$

From (3.43) and

$$d\log\tau_{k,l} = \sum_{i=1}^{n} T_{k,l}(H_i), \quad d\log\tau_{k,-l} = \sum_{i=1}^{n} T_{k,-l}(H_i) \qquad (k,l = 1,\dots,n+3,\ k\neq l), \tag{3.44}$$

we obtain

$$\frac{\tau_{n+1,n+2}\,\tau_{n+3,-(n+1)}}{\tau_0\,\tau_{n+2,n+3}} = \left(d_{n+1} - b_{n+1}\frac{d_{n+2}}{b_{n+2}}\right) \times \prod_{i=1}^{n} t_i^{1/(n+1)} \prod_{i=1}^{n+2}\prod_{j=1,j\neq i}^{n+2} (t_i-t_j)^{-1/\{2(n+1)(n+2)\}}. \tag{3.45}$$

Hence the Hirota-Miwa-equation

$$\begin{aligned}
&\tau_{n+1,n+2}\,\tau_{n+3,-(n+1)} - \tau_{n+1,n+3}\,\tau_{n+2,-(n+1)} \\
&\quad = \theta_{n+1}\prod_{i=1}^{n} t_i^{1/(n+1)} \prod_{i=1}^{n+2}\prod_{j=1,j\neq i}^{n+2} (t_i-t_j)^{-1/\{2(n+1)(n+2)\}}\, \tau_0\tau_{n+2,n+3}
\end{aligned} \tag{3.46}$$

is obtained by the action of the transformation r_{n+1} on the both sides of (3.45).

For the other indexies $i, j, k = 1, \dots, n+3$ with i, j, k mutually distinct, the Hirota-Miwa equations are obtained in a similar way.

Theorem 3.5. *For any distinct $i, j, k = 1, \dots, n+3$, we have the following Hirota-Miwa equations*:

$$F_k^{ij} \tau_0 \, \tau_{i,j} = \tau_{i,k} \, \tau_{j,-k} - \tau_{j,k} \, \tau_{i,-k}, \tag{3.47}$$

where

$$\begin{aligned} F_k^{ij} &= \theta_k \, (t_i - t_j)^{1/2} (t_i - t_k)^{-1/2} (t_j - t_k)^{-1/2} \\ &\qquad \times \prod_{l=1, l \neq k}^{n+2} (t_k - t_l)^{1/(n+1)} \prod_{l_1=1}^{n+2} \prod_{l_2=1, l_2 \neq l_1}^{n+2} (t_{l_1} - t_{l_2})^{-1/\{2(n+1)(n+2)\}}, \\ F_j^{i,n+3} &= \theta_j \prod_{k=1, k \neq j}^{n+2} (t_j - t_k)^{1/(n+1)} \prod_{k=1}^{n+2} \prod_{l=1, l \neq k}^{n+2} (t_k - t_l)^{-1/\{2(n+1)(n+2)\}}, \\ F_{n+3}^{ij} &= \theta_{n+3} \, (t_i - t_j)^{1/2} \prod_{k=1}^{n+2} \prod_{l=1, l \neq k}^{n+2} (t_k - t_l)^{-1/\{2(n+1)(n+2)\}}. \end{aligned} \tag{3.48}$$

3.4 Bilinear differential equations

In this section, we present the bilinear differential equations for the τ-functions τ_0 and $\tau_1 = \tau_{n+1,n+2}$. We set

$$\widehat{H}_i = \sum_{j=1, j \neq i}^{n+2} \frac{t_i(t_i - 1)}{t_i - t_j} \left(\mathrm{tr} A_i A_j - \frac{1}{2} \, \theta_i \theta_j \right) \quad (i = 1, \dots, n) \tag{3.49}$$

and

$$\widehat{H}_i^* = T_{n+1,n+2}(\widehat{H}_i) = \widehat{H}_i - \mathrm{tr} A_i R_{n+1,n+2} + \frac{\theta_i}{2} \quad (i = 1, \dots, n). \tag{3.50}$$

Denoting $\widehat{R} = R_{n+1,n+2}$, we have

$$\partial_{t_i}(\widehat{R}) = \frac{\widehat{R} A_i (\widehat{R} - I_2)}{t_i - 1} - \frac{(\widehat{R} - I_2) A_i \widehat{R}}{t_i} \quad (i = 1, \dots, n). \tag{3.51}$$

It follows that

$$
\begin{aligned}
&\delta_i(\widehat{H}_i) = \sum_{j=1, j\neq i}^{n+2} \frac{t_i(t_i-1)(t_i^2 - 2\,t_i t_j + t_j)}{(t_i-t_j)^2} \left(\mathrm{tr} A_i A_j - \frac{1}{2}\,\theta_i\theta_j \right), \\
&\delta_j(\widehat{H}_i) = \frac{t_i(t_i-1)t_j(t_j-1)}{(t_i-t_j)^2} \left(\mathrm{tr} A_i A_j - \frac{1}{2}\,\theta_i\theta_j \right) \quad (j=1,\dots,n,\, j\neq i), \\
&\delta_i(\widehat{H}_i - \widehat{H}_i^*) = \mathrm{tr} A_i(\widehat{R}-I_2)A_i\widehat{R} - \sum_{j=1, j\neq i}^{n+2} \frac{t_i(t_i-1)}{t_i-t_j}\,\mathrm{tr}\,[A_i, A_j]\widehat{R}, \\
&\delta_j(\widehat{H}_i - \widehat{H}_i^*) = t_j\,\mathrm{tr} A_i\widehat{R}A_j(\widehat{R}-I_2) - (t_j-1)\,\mathrm{tr} A_i(\widehat{R}-I_2)A_j\widehat{R} \\
&\qquad\qquad - \frac{t_j(t_j-1)}{t_i-t_j}\,\mathrm{tr}\,[A_i, A_j]\widehat{R} \quad (j=1,\dots,n,\, j\neq i),
\end{aligned}
\tag{3.52}
$$

where $\delta_i = t_i(t_i-1)\,\partial_i$, for each $i=1,\dots,n$. By using (3.52), we obtain

$$
\begin{aligned}
&\sum_{j=1}^{n} \frac{2}{2\,t_i t_j - t_i - t_j} \left\{ \delta_j(\widehat{H}_i + \widehat{H}_i^*) + (\widehat{H}_i - \widehat{H}_i^*)(\widehat{H}_j - \widehat{H}_j^*) \right\} \\
&\quad = -\frac{\mathrm{tr} A_i(\widehat{R}-I_2)A_i\widehat{R}}{t_i(t_i-1)} + \frac{1}{t_i(t_i-1)} \left(\mathrm{tr} A_i\widehat{R} - \frac{\theta_i}{2} \right)^2 \\
&\qquad + \sum_{j=1, j\neq i}^{n} \frac{2}{2\,t_i t_j - t_i - t_j} \Bigg\{ \left(\mathrm{tr} A_i\widehat{R} - \frac{\theta_i}{2} \right) \left(\mathrm{tr} A_j\widehat{R} - \frac{\theta_j}{2} \right) \\
&\qquad\qquad + (t_j-1)\,\mathrm{tr} A_i(\widehat{R}-I_2)A_j\widehat{R} - t_j\,\mathrm{tr} A_i\widehat{R}A_j(\widehat{R}-I_2) \Bigg\} \\
&\qquad + \sum_{j=1, j\neq i}^{n+2} \frac{1}{2\,t_i t_j - t_i - t_j} \left\{ (2\,t_j - 1)\,\mathrm{tr}\,[A_i, A_j]\widehat{R} - \mathrm{tr} A_i A_j + \frac{1}{2}\,\theta_i\theta_j \right\} \\
&\qquad + \sum_{j=1, j\neq i}^{n+2} \frac{2\,t_i - 1}{t_i - t_j} \left(\mathrm{tr} A_i A_j - \frac{1}{2}\,\theta_i\theta_j \right) \quad (i=1,\dots,n).
\end{aligned}
\tag{3.53}
$$

On the other hand, we obtain

$$
\begin{aligned}
\mathrm{tr}A_i(\widehat{R}-I_2)A_j\widehat{R} &= \left(\mathrm{tr}A_i\widehat{R}-\frac{\theta_i}{2}\right)\left(\mathrm{tr}A_j\widehat{R}-\frac{\theta_j}{2}\right)-\frac{1}{2}\,\mathrm{tr}\,[A_i,A_j]\widehat{R}\\
&\quad -\frac{1}{2}\,\mathrm{tr}A_iA_j+\frac{1}{4}\,\theta_i\theta_j \quad (j=1,\dots,n,\ j\neq i),\\
\mathrm{tr}A_i\widehat{R}A_j(\widehat{R}-I_2) &= \left(\mathrm{tr}A_i\widehat{R}-\frac{\theta_i}{2}\right)\left(\mathrm{tr}A_j\widehat{R}-\frac{\theta_j}{2}\right)+\frac{1}{2}\,\mathrm{tr}\,[A_i,A_j]\widehat{R}\\
&\quad -\frac{1}{2}\,\mathrm{tr}A_iA_j+\frac{1}{4}\,\theta_i\theta_j \quad (j=1,\dots,n,\ j\neq i),\\
\mathrm{tr}A_i(\widehat{R}-I_2)A_i\widehat{R} &= \left(\mathrm{tr}A_i\widehat{R}-\frac{\theta_i}{2}\right)^2-\frac{\theta_i^2}{4\,t_i(t_i-1)}
\end{aligned}
\tag{3.54}
$$

and

$$
\begin{aligned}
&\mathrm{tr}[A_i,A_{n+1}]\widehat{R}+\mathrm{tr}A_iA_{n+1}=\theta_{n+1}\,\mathrm{tr}A_i\widehat{R},\\
&\mathrm{tr}[A_i,A_{n+2}]\widehat{R}-\mathrm{tr}A_iA_{n+2}=\theta_{n+2}\,\mathrm{tr}A_i(\widehat{R}-I_2)
\end{aligned}
\tag{3.55}
$$

by direct computations for each $i=1,\dots,n$. From (3.53), (3.54) and (3.55), the following differential equations are obtained:

$$
\begin{aligned}
&\sum_{j=1}^{n}\frac{2}{2\,t_it_j-t_i-t_j}\left\{\delta_j(\widehat{H}_i+\widehat{H}_i^*)+(\widehat{H}_i-\widehat{H}_i^*)(\widehat{H}_j-\widehat{H}_j^*)\right\}\\
&\quad=\left(\frac{\theta_{n+1}}{t_i}+\frac{\theta_{n+2}}{t_i-1}\right)(\widehat{H}_i-\widehat{H}_i^*)+\frac{2\,t_i-1}{t_i(t_i-1)}\,\widehat{H}_i+\frac{\theta_i^2}{4\,t_i(t_i-1)}\\
&\qquad\qquad (i=1,\dots,n).
\end{aligned}
\tag{3.56}
$$

By substituting

$$
\widehat{H}_i=\delta_i\log\tau_0+\widehat{C}_i,\quad \widehat{H}_i^*=\delta_i\log\tau_1+\widehat{C}_i^*,
\tag{3.57}
$$

where

$$
\widehat{C}_i=\sum_{j=1,j\neq i}^{n+2}\frac{t_i(t_i-1)}{t_i-t_j}\left(C_{ij}+\frac{1}{2}\,\theta_i\theta_j\right),\quad \widehat{C}_i^*=T_{n+1,n+2}(\widehat{C}_i)
\tag{3.58}
$$

into (3.56), we obtain the bilinear differential equations for the τ-functions τ_0 and τ_1.

Theorem 3.6. *The τ-functions τ_0 and τ_1 satisfy the following bilinear differential equations*:

$$
\begin{aligned}
&\sum_{j=1}^{n}\frac{2}{2\,t_it_j-t_i-t_j}\left\{D_i^*D_j^*\,\tau_0\cdot\tau_1+F_j^iD_j^*\,\tau_0\cdot\tau_1\right\}+F^{i,0}D_i^*\,\tau_0\cdot\tau_1\\
&\quad-\frac{2\,t_i-1}{t_i(t_i-1)}\,\delta_i(\tau_0)\cdot\tau_1+F^{i,1}\tau_0\cdot\tau_1=0 \quad (i=1,\dots,n),
\end{aligned}
\tag{3.59}
$$

where

$$
\begin{aligned}
F^i_j &= \widehat{C}_i - \widehat{C}^*_i,\\
F^{i,0} &= \sum_{j=1}^{n} \frac{2\,(\widehat{C}_i - \widehat{C}^*_i)}{2\,t_i t_j - t_i - t_j} - \frac{\theta_{n+1}}{t_i} - \frac{\theta_{n+2}}{t_i - 1},\\
F^{i,1} &= \sum_{j=1}^{n} \frac{2}{2\,t_i t_j - t_i - t_j} \left\{ \delta_j(\widehat{C}_i + \widehat{C}^*_i) + (\widehat{C}_i - \widehat{C}^*_i)(\widehat{C}_j - \widehat{C}^*_j) \right\}\\
&\quad - \left(\frac{\theta_{n+1}}{t_i} + \frac{\theta_{n+2}}{t_i - 1} \right) (\widehat{C}_i - \widehat{C}^*_i) - \frac{2\,t_i - 1}{t_i(t_i-1)}\,\widehat{C}_i - \frac{\theta_i^2}{4\,t_i(t_i-1)}
\end{aligned}
\tag{3.60}
$$

and D^*_i stands for the Hirota derivative with respect to the derivation δ_i.

4 Garnier system

We consider rational functions in a_j, b_j, c_j, d_j $(j = 1, \ldots, n+2)$ defined as

$$
\begin{aligned}
q_i &= \frac{t_i b_i}{b_\infty} && (i = 1, \ldots, n),\\
p_i &= \frac{b_\infty}{t_i} \left\{ \frac{a_i}{b_i} + (t_i - 1)\,\frac{a_{n+1}}{b_{n+1}} - t_i\,\frac{a_{n+2}}{b_{n+2}} \right\} && (i = 1, \ldots, n),\\
x_i &= \frac{t_i}{t_i - 1} && (i = 1, \ldots, n),
\end{aligned}
\tag{4.1}
$$

where $b_\infty = \sum_{j=1}^{n+2} t_j b_j$. Let $\{\, , \,\}$ be the Poisson bracket defined by

$$
\{\varphi, \psi\} = \sum_{j=1}^{n} \left(\frac{\partial \varphi}{\partial p_j}\frac{\partial \psi}{\partial q_j} - \frac{\partial \varphi}{\partial q_j}\frac{\partial \psi}{\partial p_j} \right). \tag{4.2}
$$

Also let $\bar{d}$ be an exterior differentiation with respect to $x_1, \ldots, x_n$. Then we have

Proposition 4.1 ([1]). *The independent and dependent variables q_i, p_i, x_i $(i = 1, \ldots, n)$ defined by (4.1) satisfy the Garnier system*

$$
\bar{d} q_i = \sum_{j=1}^{n} \{\bar{H}_j, q_i\}\, dx_j, \quad \bar{d} p_i = \sum_{j=1}^{n} \{\bar{H}_j, p_i\}\, dx_j \tag{4.3}
$$

with the Hamiltonians

$$
-(x_i - 1)^2\, \bar{H}_i = T_{n+3,-(n+1)}(H_i) \quad (i = 1, \ldots, n). \tag{4.4}
$$

Here we remark

$$\bar{H}_i = K_i + \sum_{j=1, j\neq i}^{n+2} \frac{\bar{C}_{ij}}{x_i - x_j} \quad (i = 1, \ldots, n), \tag{4.5}$$

where

$$\begin{aligned} \bar{C}_{ij} &= T_{n+3,-(n+1)}(C_{ij}) + \theta_i\theta_j \quad (j = 1, \ldots, n), \\ \bar{C}_{i\,n+1} &= T_{n+3,-(n+1)}(C_{i\,n+1}) + \theta_i(\theta_{n+1} - 1), \\ \bar{C}_{i\,n+2} &= -\sum_{j=1, j\neq i}^{n+2} T_{n+3,-(n+1)}(C_{ij}) + \theta_i(\theta_i + \theta_{n+3} + 2\rho + 1) \end{aligned} \tag{4.6}$$

and K_i is given by (1.9).

In this section, we show that the Garnier system has affine Weyl group symmetry of type $B^{(1)}_{n+3}$. We also show that the τ-functions for the Garnier system, formulated on the root lattice $Q(C_{n+3})$, satisfy Toda equations, Hirota-Miwa equations and bilinear differential equations.

4.1 Affine Weyl group symmetries

The transformations σ_k, r_l and T_μ given in Section 2 can be lifted to the birational canonical transformations of the variables q_i, p_i, x_i $(i = 1, \ldots, n)$ which is already known in [7, 8]. In this section, we formulate the action of those transformations as realization of affine Weyl group.

Denote the parameter by

$$\begin{aligned} &\varepsilon_1 = \theta_{n+1}, \quad \varepsilon_2 = \theta_{n+2}, \quad \varepsilon_3 = \theta_{n+3} + 1, \\ &\varepsilon_j = \theta_{j-3} \quad (j = 4, \ldots, n+3). \end{aligned} \tag{4.7}$$

Then the group of symmetries for the Garnier system is generated by the transformations s_k $(k = 0, 1, \ldots, n+3)$ which act on ε_j $(j = 1, \ldots, n+3)$ as follows:

$$\begin{aligned} &s_0(\varepsilon_1) = 1 - \varepsilon_2, \quad s_0(\varepsilon_2) = 1 - \varepsilon_1, \quad s_0(\varepsilon_j) = \varepsilon_j \quad (j \neq 1, 2), \\ &s_k(\varepsilon_j) = \varepsilon_{\sigma_k(j)} \quad (k = 1, \ldots, n+2), \\ &s_{n+3}(\varepsilon_j) = (-1)^{\delta_{j\,n+3}}\varepsilon_j \quad (j \neq n+3). \end{aligned} \tag{4.8}$$

We describe the action of s_k on the variables q_i, p_i, x_i $(i = 1, \ldots, n)$.

$$s_0(q_j) = \frac{p_j(q_jp_j - \varepsilon_{j+3})}{Q_1(Q_1 + \varepsilon_3)}, \quad s_0(q_jp_j) = \varepsilon_{j+3} - q_jp_j, \quad s_0(x_i) = \frac{1}{x_i}, \tag{4.9}$$

where

$$Q_1 = \sum_{l=1}^{n} q_l p_l + \frac{1}{2}\left(1 - \sum_{l=1}^{n+3} \varepsilon_l\right), \tag{4.10}$$

for $k = 0$.

$$s_1(q_j) = \frac{q_j}{x_j}, \quad s_1(p_j) = x_j p_j, \quad s_1(x_i) = \frac{1}{x_i} \tag{4.11}$$

for $k = 1$.

$$s_2(q_j) = \frac{q_j}{Q_2}, \quad s_2(p_j) = (p_j - Q_1)Q_2, \quad s_2(x_i) = \frac{x_i}{x_i - 1}, \tag{4.12}$$

where

$$Q_2 = \sum_{j=1}^{n} q_j - 1, \tag{4.13}$$

for $k = 2$.

$$\begin{aligned} &s_3(q_1) = \frac{1}{q_1}, && s_3(q_j) = -\frac{q_j}{q_1} && (j \neq 1), \\ &s_3(p_1) = -q_1 Q_1, && s_3(p_j) = -q_1 p_j && (j \neq 1), \\ &s_3(x_1) = \frac{1}{x_1}, && s_n(x_i) = \frac{x_i}{x_1} && (i \neq 1) \end{aligned} \tag{4.14}$$

for $k = 3$.

$$s_k(q_j) = q_{\sigma_{k-3}(j)}, \quad p_k(q_j) = p_{\sigma_{k-3}(j)}, \quad s_k(x_j) = x_{\sigma_{k-3}(j)} \tag{4.15}$$

for $k = 4, \ldots, n+2$.

$$\begin{aligned} &s_{n+3}(q_j) = q_j, \\ &s_{n+3}(p_n) = p_n - \frac{\varepsilon_{n+3}}{q_n}, \quad s_{n+3}(p_j) = p_j \quad (j \neq n), \\ &s_{n+3}(x_i) = x_i \end{aligned} \tag{4.16}$$

for $k = n+3$. The group generated by these s_k is isomorphic to affine Weyl group $W(B_{n+3}^{(1)})$.

Theorem 4.2. *The birational canonical transformations s_k ($k = 0, \ldots, n+3$) satisfy the fundamental relations for the generators of $W(B_{n+3}^{(1)})$*

$$\begin{aligned} &s_k^2 = 1 && (k = 0, \ldots, n+3), \\ &(s_k s_l)^2 = 1 && (k, l \neq 0, 1, 2, \ |k - l| > 1), \\ &(s_k s_{k+1})^3 = 1 && (k = 1, \ldots, n+1), \\ &(s_0 s_1)^2 = 1, && (s_0 s_2)^3 = 1, \qquad (s_{n+2} s_{n+3})^4 = 1. \end{aligned} \tag{4.17}$$

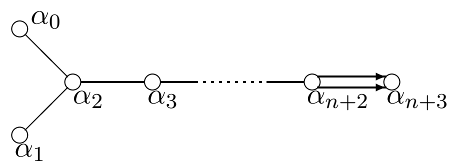

Figure 1: Dynkin diagram of type $B^{(1)}_{n+3}$

The simple affine roots of $B^{(1)}_{n+3}$ is given as

$$\begin{aligned}
&\alpha_0 = 1 - \varepsilon_1 - \varepsilon_2,\\
&\alpha_j = \varepsilon_j - \varepsilon_{j+1} \qquad (j = 1, \dots, n+2),\\
&\alpha_{n+3} = \varepsilon_{n+3}
\end{aligned} \tag{4.18}$$

and the action of s_k on α_j $(j = 0, 1, \dots, n+3)$ is described as follows.

$$s_0(\alpha_0) = -\alpha_0, \quad s_0(\alpha_2) = \alpha_0 + \alpha_2, \quad s_0(\alpha_j) = \alpha_j \quad (j \neq 0, 2) \tag{4.19}$$

for $k = 0$.

$$s_1(\alpha_1) = -\alpha_1, \quad s_1(\alpha_2) = \alpha_1 + \alpha_2, \quad s_1(\alpha_j) = \alpha_j \quad (j \neq 0, 1) \tag{4.20}$$

for $k = 1$.

$$\begin{aligned}
&s_2(\alpha_2) = -\alpha_2,\\
&s_2(\alpha_j) = \alpha_j + \alpha_2 \quad (j = 0, 1, 3),\\
&s_2(\alpha_j) = \alpha_j \qquad\quad\;\; (j \neq 0, 1, 2, 3)
\end{aligned} \tag{4.21}$$

for $k = 2$.

$$\begin{aligned}
&s_k(\alpha_k) = -\alpha_k, \quad s_k(\alpha_{k+1}) = \alpha_{k+1} + \alpha_k, \quad s_k(\alpha_{k-1}) = \alpha_{k-1} + \alpha_k,\\
&s_k(\alpha_j) = \alpha_j \quad (j \neq k, k+1, k-1)
\end{aligned} \tag{4.22}$$

for $k = 3, \dots, n+2$.

$$\begin{aligned}
&s_{n+3}(\alpha_{n+3}) = -\alpha_{n+3}, \quad s_{n+3}(\alpha_{n+2}) = \alpha_{n+2} + 2\,\alpha_{n+3},\\
&s_{n+3}(\alpha_j) = \alpha_j \quad (j \neq n+2, n+3)
\end{aligned} \tag{4.23}$$

for $k = n+3$.

Remark 4.3. *The group generated by the transformations $s_1, \dots, s_{n+2}$ is isomorphic to the symmetric group $\mathfrak{S}_{n+3}$ [1]. Furthermore, the group generated by $s_1, \dots, s_{n+3}$ is isomorphic to $W(B_{n+3})$; e.g. [5].*

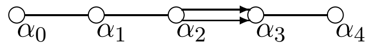

Figure 2: Dynkin diagram of type $F_4^{(1)}$

Remark 4.4. *In the only case $n = 1$, there is the following birational canonical transformation*:

$$\begin{aligned} &s_0^*(q) = q - \frac{\varepsilon_4}{p}, \quad s_0^*(p) = p, \quad s_0^*(t) = t, \\ &s_0^*(\varepsilon_j) = \varepsilon_j + \frac{1}{2}(1 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \quad (j = 1, \dots, 4). \end{aligned} \tag{4.24}$$

The transformation s_0 is generated by a composition of s_0^* and $s_1, \dots, s_4$.

But s_0^* cannot be generated by a composition of s_0, $s_1, \dots, s_4$. It follows that the group of symmetries for the Garnier system in 1-variable contains affine Weyl group $W(B_4^{(1)})$. Actually, it is known that P_{VI} has affine Weyl group symmetry of type $F_4^{(1)}$. The simple affine roots of $F_4^{(1)}$ is given by

$$\begin{aligned} &\alpha_0 = \varepsilon_1 - \varepsilon_2, \quad \alpha_1 = \varepsilon_2 - \varepsilon_3, \quad \alpha_2 = \varepsilon_3 - \varepsilon_4, \\ &\alpha_3 = \varepsilon_4, \qquad\quad \alpha_4 = \frac{1}{2}(1 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \end{aligned} \tag{4.25}$$

and s_0^*, $s_1, \dots, s_4$ act on α_j $(j = 0, 1, \dots, 4)$ as follows:

$$\begin{array}{llll} s_0^*(\alpha_4) = -\alpha_4, & s_0^*(\alpha_3) = \alpha_3 + \alpha_4, & s_0^*(\alpha_j) = \alpha_j & (j \neq 3, 4), \\ s_1(\alpha_0) = -\alpha_0, & s_1(\alpha_1) = \alpha_1 + \alpha_0, & s_1(\alpha_j) = \alpha_j & (j \neq 0, 1), \\ s_2(\alpha_1) = -\alpha_1, & s_2(\alpha_i) = \alpha_i + \alpha_1, & s_2(\alpha_j) = \alpha_j & (i = 0, 2,\ j = 3, 4), \\ s_3(\alpha_2) = -\alpha_2, & s_3(\alpha_i) = \alpha_i + \alpha_2, & s_3(\alpha_j) = \alpha_j & (i = 1, 3,\ j = 1, 4), \\ s_4(\alpha_3) = -\alpha_3, & s_4(\alpha_2) = \alpha_2 + 2\,\alpha_3, & s_4(\alpha_4) = \alpha_4 + \alpha_3, & \\ s_4(\alpha_j) = \alpha_j & & & (j = 1, 2). \end{array} \tag{4.26}$$

4.2 τ-Functions

For each solution of the Garnier system, we introduce the τ-functions $\bar{\tau}_\mu$ $(\mu \in L)$ satisfying the Pfaffian systems

$$\bar{d} \log \bar{\tau}_\mu = \sum_{i=1}^{n} T_\mu(\bar{H}_i)\, dx_i. \tag{4.27}$$

Each $\bar\tau_\mu$ is determined up to multiplicative constants. From (4.4), we can identify these $\bar\tau_\mu$ with the τ-functions for the Schlesinger system by

$$\bar\tau_0 = \tau_{n+3,-(n+1)}. \tag{4.28}$$

Hence we can apply the properties of the τ-functions τ_μ system to the Garnier system. For each $\mu \in L$, the action of the birational canonical transformations s_k on $\bar\tau_\mu$ is defined by

$$s_k(\bar\tau_\mu) = \bar\tau_{s_k(\mu)} \quad (k = 0, 1, \dots, n+3), \tag{4.29}$$

where

$$\begin{aligned} s_0(\mu) &= (1-\mu_2, 1-\mu_1, \mu_3, \dots, \mu_{n+3}),\\ s_k(\mu) &= (\mu_{(k,k+1)1}, \dots, \mu_{(k,k+1)(n+3)}) \quad (k = 1, \dots, n+2),\\ s_{n+3}(\mu) &= (\mu_1, \dots, \mu_{n+2}, -\mu_{n+3}) \end{aligned} \tag{4.30}$$

and $(k, k+1)$ stands for the adjacent transpositions. We also obtain bilinear relations which are satisfied by $\bar\tau_\mu$ formulated on the root lattice $Q(C_{n+3})$.

Theorem 4.5. *The τ-functions $\bar\tau_\mu$ $(\mu \in L)$ satisfy the Toda equations, the Hirota-Miwa equations and the bilinear differential equations given in Section 3.*

In the last, we present the following proposition.

Proposition 4.6. *For the τ-functions*

$$\bar\tau_{1,-2} = \bar\tau_{\mathbf{e}_1-\mathbf{e}_2}, \quad \bar\tau_{1,3} = \bar\tau_{\mathbf{e}_1+\mathbf{e}_3}, \quad \bar\tau_{1,-3} = \bar\tau_{\mathbf{e}_1-\mathbf{e}_3}$$

and $\bar\tau_0$, the following relations are satisfied:

$$\begin{aligned} q_i &= -\frac{1}{\varepsilon_3}\, x_i(x_i-1)\,\frac{\partial}{\partial x_i}\log\frac{\bar\tau_{1,3}}{\bar\tau_{1,-3}} + 2\,\bar X_i \quad (i = i, \dots, n),\\ q_i p_i &= -x_i\,\frac{\partial}{\partial x_i}\log\frac{\bar\tau_{1,-2}}{\bar\tau_0} + \frac{\bar\Gamma_{-1}^{j+3} - x_i\bar\Gamma_{-2}^{j+3} - (\varepsilon_1-\varepsilon_2)\bar X_i}{x_i - 1} \quad (i = 1, \dots, n), \end{aligned} \tag{4.31}$$

where

$$\bar X_i = \sum_{j=1, j\neq i}^{n+2} \frac{x_i(x_j-1)}{(n+1)(n+2)(x_i-x_j)}, \quad \bar\Gamma_{-k}^{i} = -\frac{\varepsilon_i}{2} + \frac{1-2\,\varepsilon_k}{2\,(n+1)}. \tag{4.32}$$

Proof By using (4.1), (4.7) and (4.28), we can rewrite the relations (4.31) into

$$
\begin{aligned}
q_i &= \frac{t_i}{\theta_{n+3}+1}\,\partial_i \log \frac{\tau_{2\,\mathbf{e}_{n+3}}}{\tau_0} - \sum_{j=1, j\neq i}^{n+2} \frac{2\,t_i}{(n+1)(n+2)(t_i-t_j)},\\
q_i p_i &= t_i(t_i-1)\,\partial_i \log \frac{\tau_{n+3,-(n+2)}}{\tau_{n+3,-(n+1)}} + (t_i-1)\,\mathit{\Gamma}^i_{-(n+1)} - t_i \mathit{\Gamma}^i_{-(n+2)} \\
&\qquad + \sum_{j=1, j\neq i}^{n+2} \frac{t_i(t_i-1)(\theta_{n+1}-\theta_{n+2})}{(n+1)(n+2)(t_i-t_j)} \qquad (i=1,\dots,n),
\end{aligned}
\tag{4.33}
$$

where

$$
\mathit{\Gamma}^i_{-k} = -\frac{\theta_i}{2} + \frac{1-2\,\theta_k}{2\,(n+1)}. \tag{4.34}
$$

Hence we show the relations (4.33) in the following.

We consider the Schlesinger transformations $T_{2\,\mathbf{e}_{n+3}}$ which act on the parameters as follows:

$$
T_{2\,\mathbf{e}_{n+3}}(\theta_j) = \theta_j + 2\,\delta_{j\,n+3} \quad (j=1,\dots,n+3). \tag{4.35}
$$

The action of $T_{2\mathbf{e}_{n+3}}$ on the Hamiltonians H_i $(i=1,\dots,n)$ is described as follows:

$$
T_{2\,\mathbf{e}_{n+3}}(H_i) = H_i + (\theta_{n+3}+1)\,\frac{b_i}{b_\infty} + \sum_{j=1, j\neq i}^{n+2} \frac{2\,(\theta_{n+3}+1)}{(n+1)(n+2)(t_i-t_j)}. \tag{4.36}
$$

From (3.44) and (4.36), the first relation of (4.33) is obtained. The second relation of (4.33) is obtained in a similar way. □

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